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Fixed Point Theory for k–CAR Sets

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New fixed point results are presented for maps defined on subsets of k–CAR sets. In particular our results include some of the results in the literature for hyperconvex spaces.

1. INTRODUCTION

This paper presents a variety of fixed point results for k–CAR sets in a Fréchet space E. A set $A \subseteq E$ is said to be k–CAR if there exists a continuous k–set contractive retraction from $\overline{\text{co}}(A)$ to A. Section 2 presents two fixed point results for upper semicontinuous self maps between k–CAR sets. In addition four nonlinear alternatives of Leray–Schauder type are presented for upper semicontinuous maps $F: \overline{U} \to 2^C$; here C is a k–CAR set with U open in C, and 2^C denotes the family of nonempty subsets of C. The first two alternatives discuss the case when U is arbitrary, whereas the second two discuss the special case when U is convex. From these alternatives we are able to present a fixed point result of Furi–Pera type for upper semicontinuous maps $F: Q \to 2^C$; here C is a k–CAR set and $Q \subseteq C$. Section 3 discusses the special case when our k–CAR set is closed and

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convex. The proofs of our results in Section 3 can be found in the literature [10–17]. However we note that *one* of the conditions in the corresponding theorems in [10, 12–15, 17] was stated incompletely (but applied correctly in applications); in [10, 12–15, 17] we assumed U was open (with the specified topology) in Q but in fact our set U must *also be open in* C (this was omitted in [10, 12–15, 17] by mistake).

For the remainder of the introduction we gather together some definitions which will be needed later. Let $X = (X, d)$ be a metric space and let Ω_X be the bounded subsets of X. The Kuratowski measure of noncompactness $\alpha\colon \Omega_X \to [0, \infty)$ is defined by (here $B \in \Omega_X$),

$$\alpha(B) = \inf\left\{r > 0 \colon B \subseteq \cup_{i=1}^{n} B_i \text{ and } \operatorname{diam}(B_i) \le r\right\}.$$

Let S be a nonempty subset of X and let $G\colon S \to 2^X$. Then $G\colon S \to 2^X$ is (i) k–set contractive ($k \ge 0$) if $\alpha(G(W)) \le k\alpha(W)$ for all nonempty bounded sets W of S, (ii) condensing if G is 1–set contractive and $\alpha(G(W)) < \alpha(W)$ for all bounded sets W of S with $\alpha(W) \ne 0$.

2. k–CAR SETS

In this section we present nonlinear alternatives of Leray–Schauder type for k–CAR sets. From these alternatives we are able to deduce new fixed point theorems.

Throughout this section E will be a Fréchet space.

DEFINITION 2.1. A closed subset A of E is said to be k–CAR ($k \ge 0$) if there exists a continuous k–set contractive retraction R from $\overline{\text{co}}(A)$ to A; here $\overline{\text{co}}(A)$ denotes the closed convex hull of A.

EXAMPLE 2.1. It is well known [1, 2, 9] that a hyperconvex [8, 9] subset A of a metric space is a 1-CAR set (in fact the retraction R is nonexpansive). Recall a metric space (X, d) is hyperconvex if $\cap_\alpha B(x_\alpha, r_\alpha) \ne \emptyset$ for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in X for which $d(x_\alpha, x_\beta) \le r_\alpha + r_\beta$.

THEOREM 2.1. *Let E be a Fréchet space, let C be a closed k–CAR ($0 \le k \le 1$) subset of E, and let U be a relatively open subset of C with $x_0 \in U$. In addition assume $F\colon \overline{U} \to K(C)$ is an upper semicontinuous condensing map with $F(\overline{U})$ a bounded set in C; here $K(C)$ denotes the family of nonempty, closed, acyclic subsets of C and $\overline{U}$ denotes the closure of U in C. Also suppose the following two conditions hold:*

$$x \notin \lambda F x + (1 - \lambda)\{x_0\} \qquad \text{for all } x \in \partial U \tag{2.1}$$
$$(\text{the boundary of } U \text{ in } C) \quad \text{and} \quad \lambda \in (0, 1),$$

$$x \notin \lambda F y + (1 - \lambda)\{x_0\} \qquad \text{for all } x \in \overline{\text{co}}(C)\backslash C, \tag{2.2}$$
$$\lambda \in (0, 1] \quad \text{and} \quad y \in U.$$

Then F has a fixed point in $\overline{U}$.

Proof. Suppose F has no fixed points in ∂U. Let

$$H = \{x \in \overline{U} : x \in \lambda Fx + (1-\lambda)\{x_0\} \text{ for some } \lambda \in [0,1]\}.$$

Notice $H \neq \emptyset$ is closed since F is upper semicontinuous. Note also that $H \cap \partial U = \emptyset$. Thus there exists a continuous function $\mu \colon \overline{U} \to [0,1]$ with $\mu(H) = 1$ and $\mu(\partial U) = 0$. Define the mapping $J \colon C \to K(\overline{\text{co}}(C))$ by

$$J(x) = \begin{cases} \mu(x)F(x) + (1-\mu(x))\{x_0\} \equiv G(x), & x \in \overline{U} \\ \{x_0\}, & x \in C \backslash \overline{U}. \end{cases}$$

We first show $J \colon C \to K(\overline{\text{co}}(C))$ is upper semicontinuous. To see this let V be a closed subset of $\overline{\text{co}}(C)$. Then

$$\begin{aligned} &\{x \in C : J(x) \cap V \neq \emptyset\} \\ &\quad = \{x \in \overline{U} : G(x) \cap V \neq \emptyset\} \cup \{x \in C \backslash \overline{U} : \{x_0\} \cap V \neq \emptyset\}. \end{aligned}$$

There are two cases to consider, namely $x_0 \in V$ and $x_0 \notin V$.

Case (i). $x_0 \notin V$.
Then

$$\{x \in C : J(x) \cap V \neq \emptyset\} = \{x \in \overline{U} : G(x) \cap V \neq \emptyset\} = G^{-1}(V),$$

which is closed in $\overline{U}$ (so closed in C) since $G \colon \overline{U} \to K(C)$ is upper semicontinuous.

Case (ii). $x_0 \in V$.

Then

$$\begin{aligned} \{x \in C : J(x) \cap V \neq \emptyset\} &= \{x \in \overline{U} : G(x) \cap V \neq \emptyset\} \cup (C \backslash \overline{U}) \\ &= \{x \in \overline{U} : G(x) \cap V \neq \emptyset\} \cup (C \backslash U). \end{aligned}$$

To see this we need only show that if $x \in C \backslash U$ then

$$x \in \{y \in \overline{U} : G(y) \cap V \neq \emptyset\} \cup (C \backslash \overline{U}).$$

This is immediate if $x \in C \backslash U$ and $x \in C \backslash \overline{U}$ also. So it remains to consider the case when $x \in C \backslash U$ and

$$x \notin C \backslash \overline{U} = C \backslash (U \cup \partial U) = (C \backslash U) \cap (C \backslash \partial U),$$

i.e., when $x \in C \backslash U$ and $x \notin (C \backslash \partial U)$. In this case $G(x) = x_0 \in V$, so $x \in \{y \in \overline{U} : G(y) \cap V \neq \emptyset\}$. Consequently

$$\{x \in C : J(x) \cap V \neq \emptyset\} = G^{-1}(V) \cup (C \backslash U),$$

which is closed in C.

Next notice $J: C \to K(\overline{\text{co}}(C))$ is a condensing map with $J(C)$ a bounded set in $\overline{\text{co}}(C)$. To see this note

$$J(A) \subseteq \overline{\text{co}}(F(A \cap \overline{U}) \cup \{x_0\})$$

for any subset A of C, so if A is a bounded subset of C with $\alpha(A) > 0$ then

$$\alpha(J(A)) \le \alpha(\overline{\text{co}}(F(A \cap \overline{U}))) = \alpha(F(A \cap \overline{U})) \le \alpha(F(A)) < \alpha(A).$$

Next let

$$R: \overline{\text{co}}(C) \to C$$

be the continuous k–set retraction which is guaranteed since C is a k–CAR set. Let us look at the map

$$\theta = JR: \overline{\text{co}}(C) \to K(\overline{\text{co}}(C)).$$

Notice $\theta: \overline{\text{co}}(C) \to K(\overline{\text{co}}(C))$ is upper semicontinuous, condensing, and $\theta(\overline{\text{co}}(C))$ is a bounded set in $\overline{\text{co}}(C)$. To see condensing let A be a bounded subset of $\overline{\text{co}}(C)$ with $\alpha(A) > 0$. If $\alpha(R(A)) > 0$ then

$$\alpha(\theta(A)) = \alpha(JR(A)) < \alpha(R(A)) \le k\alpha(A) \le \alpha(A),$$

whereas if $\alpha(R(A)) = 0$ then

$$\alpha(\theta(A)) = \alpha(JR(A)) \le \alpha(R(A)) = 0 < \alpha(A).$$

Now a result in [5, 16] guarantees that there exists $x \in \overline{\text{co}}(C)$ with $x \in \theta(x)$. There are two cases to consider, namely $x \in C$ and $x \notin C$.

Case (i). $x \in C$.
Then

$$x \in \theta(x) = JR(x) = J(x).$$

If $x \in C \backslash \overline{U}$ we obtain a contradiction since then $x \in J(x) = \{x_0\}$, but $x_0 \in U$. On the other hand if $x \in U$ then

$$x \in J(x) = \mu(x)F(x) + (1 - \mu(x))\{x_0\}.$$

That is, $x \in \lambda F(x) + (1 - \lambda)\{x_0\}$, where $0 \le \lambda = \mu(x) \le 1$. Consequently $x \in H$, so $\mu(x) = 1$. Thus $x \in F(x)$, which is the desired conclusion.

Case (ii). $x \in \overline{\text{co}}(C) \backslash C$.
Then $R(x) = y$, say, with $y \in C$. If $y \in C \backslash \overline{U}$ we obtain a contradiction since then $x \in \theta(x) = J(y) = \{x_0\}$, but $x_0 \in U$ whereas $x \in \overline{\text{co}}(C) \backslash C$. Finally suppose $y \in U$. Then

$$x \in \theta(x) = J(y) = \mu(y)F(y) + (1 - \mu(y))\{x_0\}.$$

Note this contradicts (2.2) if $\mu(y) > 0$, whereas if $\mu(y) = 0$ we also have a contradiction since $x_0 \in U$ but $x \in \overline{\text{co}}(C) \backslash C$. ∎

Remark 2.1. From the proof of Theorem 2.1 it is clear that (2.2) can be replaced in Theorem 2.1 by the less restrictive condition

$$x \notin \lambda F y + (1-\lambda)\{x_0\} \quad \text{for all } x \in \overline{co}(C) \backslash C,\ \lambda \in (0,1] \quad \text{and } y = R(x) \in U; \text{here } R\colon \overline{\text{co}}(C) \to C \text{ is the continuous } k\text{-set retraction guaranteed since } C \text{ is a } k\text{-CAR set.} \tag{2.3}$$

Remark 2.2. It is easy to see that in Theorem 2.1, $F\colon \overline{U} \to K(C)$ upper semicontinuous can be replaced by $F\colon \overline{U} \to K(C)$ a closed map (i.e., has closed graph).

Remark 2.3. In Theorem 2.1 the upper semicontinuous map $F\colon \overline{U} \to K(C)$ can be replaced by an approximable closed map [16].

Remark 2.4. if C is a convex subset of E then Theorem 2.1 reduces to the standard nonlinear alternative of Leray–Schauder type found in [16]; notice (2.2) is trivially satisfied in this case since $\overline{\text{co}}(C) \backslash C = \emptyset$.

Our next result improves Theorem 2.1 when $\partial U = \emptyset$.

THEOREM 2.2. *Let E be a Fréchet space and let C be a nonempty closed k–CAR* $(0 \le k \le 1)$ *subset of E. Suppose $F\colon C \to K(C)$ is an upper semicontinuous condensing map with $F(C)$ a bounded set in C. Then F has a fixed point in C.*

Proof. The proof follows from the ideas in Theorem 2.1 with $U = C$, so $\partial U = \emptyset$. In this case we let $J = F$ so $J\colon C \to K(C)$, and we let $R\colon \overline{\text{co}}(C) \to C$ be the continuous k–set retraction which is guaranteed since C is a k–CAR set. Notice $\theta = FR\colon \overline{\text{co}}(C) \to K(C)$ is a upper semicontinuous, condensing, map and $\theta(\overline{\text{co}}(C))$ is a bounded set in C. Now [5, 16] guarantees that there exists $x \in \overline{\text{co}}(C)$ with $x \in \theta(x)$. In fact since $\theta\colon \overline{\text{co}}(C) \to K(C)$ we must have $x \in C$ and as a result $x \in FR(x) = F(x)$. ∎

Remark 2.5. When C is hyperconvex and $F\colon C \to C$ is continuous (and single valued) then Theorem 2.2 reduces to a result in [4, 9].

Essentially the same reasoning as in Theorems 2.1 and 2.2 yields the following two results.

THEOREM 2.3. *Let E be a Fréchet space, let C be a closed k–CAR* $(k \ge 0)$ *subset of E, and let U be a relatively open subset of C with $x_0 \in U$. In addition assume $F\colon \overline{U} \to K(C)$ is an upper semicontinuous, β–set contractive* $(\beta \ge 0)$ *map with $F(\overline{U})$ a bounded set in C and $k\beta < 1$. Also suppose* (2.1) *and* (2.2) *hold. Then F has a fixed point in $\overline{U}$.*

THEOREM 2.4. *Let E be a Fréchet space and let C be a nonempty closed k–CAR ($k \geq 0$) subset of E. Suppose $F: C \to K(C)$ is an upper semicontinuous, β–set contractive ($\beta \geq 0$) map with $F(C)$ a bounded set in C and $k\beta < 1$. Then F has a fixed point in C.*

Next we show how the ideas in Theorem 2.2 can be used to improve considerably Theorems 2.1 and 2.3 when $\overline{U}$ is convex.

THEOREM 2.5. *Let E be a Fréchet space, let C be a closed k–CAR ($0 \leq k \leq 1$) subset of E, and let U be a relatively open subset of C, $0 \in U$ and $\overline{U}$ convex. In addition assume $F: \overline{U} \to K(C)$ is an upper semicontinuous condensing map with $F(\overline{U})$ a bounded set in C. Also suppose*

$$x \notin \lambda Fx \qquad \textit{for all } x \in \partial U \quad \textit{and} \quad \lambda \in (0,1). \tag{2.4}$$

Then F has a fixed point in $\overline{U}$.

Proof. Let $r: E \to \overline{U}$ be given by

$$r(x) = \frac{x}{\max\{1, g(x)\}},$$

where g is the Minkowski functional on $\overline{U}$; i.e., $g(x) = \inf\{\alpha > 0 : x \in \alpha\overline{U}\}$. Recall [10] that $r: E \to \overline{U}$ is a 1-set contractive map. Now let $J = Fr: C \to K(C)$ (note J is a upper semicontinuous condensing map with $J(C)$ a bounded set in C). Now Theorem 2.2 guarantees that there exists

$$x \in C \qquad \text{with } x \in J(x) = Fr(x).$$

Let $z = r(x)$. Then we have immediately that $z \in r(F(z))$; i.e., $z = r(w)$ for some $w \in F(z)$. There are two cases to consider, namely $w \in \overline{U}$ or $w \in C \backslash \overline{U}$. If $w \in C \backslash \overline{U}$ then

$$z = r(w) = \frac{w}{g(w)} \in \partial U \qquad \text{and} \qquad z \in \lambda F(z)$$

$$\text{where } \lambda = \frac{1}{g(w)} \in (0,1).$$

This contradicts (2.4). If $w \in \overline{U}$ then

$$z = r(w) = w \in \overline{U} \qquad \text{and} \qquad z \in F(z),$$

and we are finished. ∎

Remark 2.6. In Theorem 2.5 if $F(\partial U) \subseteq \overline{U}$, then (2.4) is satisfied.

THEOREM 2.6. *Let E be a Fréchet space, let C be a closed k–CAR ($k \geq 0$) subset of E, and let U be a relatively open subset of C, $0 \in U$, and $\overline{U}$ convex. In addition assume $F: \overline{U} \to K(C)$ is an upper semicontinuous, β–set contractive ($\beta \geq 0$) map with $F(\overline{U})$ a bounded set in C and $k\beta < 1$. Also suppose* (2.4) *holds. Then F has a fixed point in $\overline{U}$.*

Our next theorem is a Furi–Pera type result [7, 10] for k–CAR sets.

THEOREM 2.7. *Let $E = (E, d)$ be a Fréchet space, let C be a closed k–CAR ($k \geq 0$) subset of E, let Q be a closed, convex, proper subset of C with $0 \in Q$, and let*

$$U_i = \left\{ x \in E : d(x, Q) < \frac{1}{i} \right\} \subseteq C \qquad \text{for } i \text{ sufficiently large.}$$

Suppose $F: Q \to K(C)$ is a upper semicontinuous, compact map with the following satisfied:

$$\begin{aligned}&\text{if } \{(x_j, \lambda_j)\}_1^\infty \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x, \lambda)\\ &\text{with } x \in \lambda F(x) \text{ and } 0 \leq \lambda < 1, \text{ then there exists } j_0 \in \{1, 2, \ldots.\}\\ &\text{with } \{\lambda_j F(x_j)\} \subseteq Q \text{ for each } j \geq j_0; \text{ here } \partial Q \text{ denotes the}\\ &\text{boundary of } Q \text{ in } C.\end{aligned} \tag{2.5}$$

Then F has a fixed point in Q.

Proof. Let $r: E \to Q$ be a continuous retraction chosen so that $r(z) \in \partial Q$ for $z \in E \backslash Q$ (see [10]). Consider

$$B = \{x \in C : x \in Fr(x)\}.$$

Now since $Fr: C \to K(C)$ is an upper semicontinuous, compact map then Theorem 2.4 guarantees that $B \neq \emptyset$. Of course B is closed and also compact since $B \subseteq Fr(B) \subseteq F(Q)$. It remains to show $B \cap Q \neq \emptyset$. To do so we argue by contradiction. Suppose $B \cap Q = \emptyset$. Then since B is compact and Q is closed there exists $\delta > 0$ with $\text{dist}(B, Q) > \delta$. Choose $m \in \{1, 2, \ldots\}$ such that $1 < \delta m$ and $U_i \subseteq C$ for $i \in \{m, m+1, \ldots\}$; here

$$U_i = \left\{ x \in E : d(x, Q) < \frac{1}{i} \right\} \qquad \text{for } i \in \{m, m+1, \ldots\}.$$

Fix $i \in \{m, m+1, \ldots\}$. Since $\text{dist}(B, Q) > \delta$ then $B \cap \overline{U_i} = \emptyset$. Also notice U_i is open in C, U_i is convex, $0 \in U_i$, and $Fr: \overline{U_i} \to K(C)$ is a upper semicontinuous, compact map. Now $B \cap \overline{U_i} = \emptyset$ and Theorem 2.6 (applied with $U = U_i$, $C = C$, and $F = Fr$; note (2.4) cannot hold) guarantees that there exists $(y_i, \lambda_i) \in \partial U_i \times (0, 1)$ (∂U_i denotes the boundary of U_i in C) with $y_i \in \lambda_i Fr(y_i)$. We can do this argument for each $i \in \{m, m+1, \ldots\}$. Notice in particular since $y_i \in \partial U_i$ that

$$\{\lambda_i Fr(y_i)\} \not\subseteq Q \qquad \text{for each } i \in \{m, m+1, \ldots\}. \tag{2.6}$$

Now consider

$$D = \{x \in C : x \in \lambda Fr(x) \text{ for some } \lambda \in [0, 1]\}.$$

It is immediate that D is closed and compact (since $D \subseteq \mathrm{co}(Fr(D) \cup \{0\}) \subseteq \mathrm{co}(F(Q) \cup \{0\})$). This together with

$$d(y_j, Q) = \frac{1}{j}, \quad |\lambda_j| \le 1 \qquad \text{for } j \in \{m, m+1, \ldots\}$$

implies that we may assume without loss of generality that

$$\lambda_j \to \lambda^\star \qquad \text{and} \qquad y_j \to y^\star \in \overline{Q} \cap \overline{C \backslash Q} = \partial Q.$$

Also since $y_j \in \lambda_j Fr(y_j)$ we have that $y^\star \in \lambda^\star Fr(y^\star)$ (recall $F: Q \to K(C)$ is upper semicontinuous). If $\lambda^\star = 1$ then $y^\star \in Fr(y^\star)$ which contradicts $B \cap Q = \emptyset$. Thus $0 \le \lambda^\star < 1$. But in this case (2.5), with

$$x_j = r(y_j) \in \partial Q \qquad \text{and} \qquad x = y^\star = r(y^\star),$$

implies that there exists $j_0 \in \{1, 2, \ldots\}$ with $\{\lambda_j Fr(y_j)\} \subseteq Q$ for each $j \ge j_0$. This contradicts (2.6). Thus $B \cap Q \ne \emptyset$; i.e., there exists $x \in Q$ with $x \in Fr(x) = F(x)$. ∎

Remark 2.7. If E in Theorem 2.7 is a Hilbert space then one could replace in Theorem 2.7 $F: Q \to K(C)$ a compact map with $F: Q \to K(C)$ a condensing map with $F(Q)$ a bounded set in C provided $0 \le k \le 1$ (or with $F: Q \to K(C)$ a β–set contractive map with $F(Q)$ a bounded set in C and $k\beta < 1$). The proof is exactly the same as in Theorem 2.7 except in this case $r: E \to Q$ is defined by $r(x) = P_Q(x)$; i.e., r is the nearest point projection on Q (recall r is nonexpansive). Of course the result in Theorem 2.5 holds for certain convex sets Q in a Fréchet space where there is a nearest point projection that is nonexpansive (or more generally 1-set contractive).

3. CLOSED, CONVEX SETS

In this section we will assume the k–CAR sets in Section 2 are closed, convex sets. Some of the results in this section may be found in [10–17]; we note that *one* of the conditions in the corresponding theorems in [10, 12–15, 17] was stated incompletely (but applied correctly in the applications). The proof of our first result can be found in [16] (see Theorem 2.1 of this paper also).

THEOREM 3.1. *Let E be a Fréchet space, let Q be a closed subset of E, let C be a closed convex subset of E with $Q \subseteq C$, and let U be a relatively open subset of Q with $x_0 \in U$. In addition assume $F: \overline{U} \to K(C)$ is an upper semicontinuous, condensing map with $F(\overline{U})$ a bounded set in C; here*

$\overline{U}$ denotes the closure of U in Q (which equals the closure of U in C). Also suppose the following condition is satisfied:

$$x \notin \lambda Fx + (1-\lambda)\{x_0\} \qquad \text{for all } x \in \partial_C U \qquad (3.1)$$
$$(\text{the boundary of } U \text{ in } C) \quad \text{and} \quad \lambda \in [0,1).$$

Then F has a fixed point in $\overline{U}$.

Remark 3.1. In [10] we stated incorrectly that (3.1) had to hold for $x \in \partial_Q U$. As we will see it should have been $x \in \partial_C U$. We note $\partial_Q U \subseteq \partial_C U$ since

$$\partial_Q U = \overline{U} \backslash U \qquad \text{and} \qquad \partial_C U = \overline{U} \backslash (\operatorname{int}_C U),$$

and $\operatorname{int}_C U \subseteq U$. Of course if U is also open in C (which is what we had in mind in [10]) or if $Q = C$ then $\partial_Q U = \partial_C U$.

Remark 3.2. Notice (3.1) guarantees that $x_0 \notin \partial_C U$ so $x_0 \in \operatorname{int}_C U$.

Proof. The proof is exactly the same as in [10]; for completeness we include it here. Suppose F has no fixed points in $\partial_C U$. Let

$$H = \{x \in \overline{U} : x \in \lambda Fx + (1-\lambda)\{x_0\} \text{ for some } \lambda \in [0,1]\}.$$

Now $H \neq \emptyset$ is closed, $H \cap \partial_C U = \emptyset$, and so there exists a continuous function $\mu: \overline{U} \to [0,1]$ with $\mu(H) = 1$ and $\mu(\partial_C U) = 0$. Define the mapping $J: C \to K(C)$ by

$$J(x) = \begin{cases} \mu(x)F(x) + (1-\mu(x))\{x_0\} \equiv G(x), & x \in \overline{U} \\ \{x_0\}, & x \in C \backslash \overline{U}. \end{cases}$$

We first show $J: C \to K(C)$ is upper semicontinuous. Let V be a closed subset of C. Then

$$\{x \in C : J(x) \cap V \neq \emptyset\}$$
$$= \{x \in \overline{U} : G(x) \cap V \neq \emptyset\} \cup \{x \in C \backslash \overline{U} : \{x_0\} \cap V \neq \emptyset\}.$$

There are two cases to consider, namely $x_0 \in V$ and $x_0 \notin V$.

Case (i). $x_0 \notin V$.
Then

$$\{x \in C : J(x) \cap V \neq \emptyset\} = \{x \in \overline{U} : G(x) \cap V \neq \emptyset\} = G^{-1}(V)$$

which is closed in $\overline{U}$ (so closed in C) since $G: \overline{U} \to K(C)$ is upper semicontinuous.

Case (ii). $x_0 \in V$. Then

$$\{x \in C : J(x) \cap V \neq \emptyset\} = \{x \in \overline{U} : G(x) \cap V \neq \emptyset\} \cup (C \backslash \overline{U})$$
$$= \{x \in \overline{U} : G(x) \cap V \neq \emptyset\} \cup (C \backslash (\mathrm{int}_C\, U)).$$

To see this we need only show that if $x \in C \backslash (\mathrm{int}_C\, U)$ then

$$x \in \{y \in \overline{U} : G(y) \cap V \neq \emptyset\} \cup (C \backslash \overline{U}).$$

This is immediate if $x \in C \backslash (\mathrm{int}_C\, U)$ and $x \in C \backslash \overline{U}$ also. So it remains to consider the case when $x \in C \backslash (\mathrm{int}_C\, U)$ and

$$x \notin C \backslash \overline{U} = C \backslash ((\mathrm{int}_C\, U) \cup \partial_C U) = (C \backslash (\mathrm{int}_C\, U)) \cap (C \backslash \partial_C U),$$

i.e., when $x \in C \backslash (\mathrm{int}_C\, U)$ and $x \notin (C \backslash \partial_C U)$. In this case $G(x) = x_0 \in V$, so $x \in \{y \in \overline{U} : G(y) \cap V \neq \emptyset\}$. Consequently

$$\{x \in C : J(x) \cap V \neq \emptyset\} = G^{-1}(V) \cup (C \backslash (\mathrm{int}_C\, U)),$$

which is closed in C.

Also notice $J: C \to K(C)$ is a condensing map with $J(C)$ a bounded set in C. Now [5] guarantees that there exists $x \in C$ with $x \in J(x)$. If $x \in C \backslash \overline{U}$ we obtain a contradiction since $x_0 \notin \partial_C U$ (see Remark 3.2). Thus $x \in U$ so $x \in \mu(x) F(x) + (1 - \mu(x))\{x_0\}$. As a result $x \in H$, so $\mu(x) = 1$. ∎

Essentially the same argument as in Theorem 2.7 (or alternatively see [10]) establishes the following result of Furi–Pera type.

THEOREM 3.2. *Let $E = (E, d)$ be a Fréchet space, let C be a closed convex subset of E, let Q be a closed, convex, proper subset of C with $0 \in Q$, and let*

$$U_i = \left\{x \in E : d(x, Q) < \frac{1}{i}\right\} \subseteq C \qquad \text{for } i \text{ sufficiently large.}$$

Suppose $F: Q \to K(C)$ is a upper semicontinuous, compact map with (2.5) *holding. Then F has a fixed point in Q.*

Remark 3.3. In applications (see [10]) we usually take $C = E$ or alternatively $C = \overline{\mathrm{co}}(F(Q) \cup \overline{U_1})$; here $U_1 = \{x \in E : d(x, Q) < 1\}$.

Remark 3.4. As in Remark 2.7, if E in Theorem 3.2 is a Hilbert space then one could replace in Theorem 3.2 $F: Q \to K(C)$ a compact map with $F: Q \to K(C)$ a condensing map with $F(Q)$ a bounded set in C.

Finally in this section we discuss weakly sequentially upper semicontinuous maps. Let E be a Banach space and suppose $F: Z \subseteq E \to 2^E$ maps bounded sets into bounded sets. We call F a α w–contractive map if $0 \leq \alpha < 1$ and $w(F(X)) \leq \alpha w(X)$ for all bounded sets $X \subseteq Z$; here w is the measure of weak noncompactness [12, 14]. We say $G: E_1 \to 2^{E_2}$ (here E_1 and E_2 are Banach spaces) is weakly upper semicontinuous if the set $G^{-1}(A)$ is weakly closed in E_1 for any weakly closed set A in E_2. G is weakly sequentially upper semicontinuous if for any weakly closed set A in E_2 we have that $G^{-1}(A)$ is sequentially closed for the weak topology on E_1.

We begin with a nonlinear alternative of Leray–Schauder type for weakly sequentially upper semicontinuous maps [12, 14] (we note that *one* of the conditions in [12, 14] was incompletely stated (but applied correctly in applications)).

THEOREM 3.3. *Let Q and C be closed bounded convex subsets of a Banach space E with $Q \subseteq C$. Also assume U is a weakly open subset of Q, $0 \in U$, and $\overline{U^w}$ is a weakly compact subset of Q (here $\overline{U^w}$ denotes the weak closure of U in Q). Suppose $F: \overline{U^w} \to CK(C)$ is a weakly sequentially upper semicontinuous, α w–contractive (here $0 \leq \alpha < 1$) map; here $CK(C)$ denotes the family of nonempty convex weakly compact subsets of C. In addition assume the following condition is satisfied:*

$$x \notin \lambda F x \qquad \text{for all } x \in \partial_C U \text{ (the weak boundary of } U \text{ in } C) \text{ and } \lambda \in [0,1). \tag{3.2}$$

Then F has a fixed point in $\overline{U^w}$.

Remark 3.5. In [12–15, 17] we stated incorrectly that (3.2) had to hold for $x \in \partial_Q U$; this should have been $x \in \partial_C U$.

Remark 3.6. The condition that $\overline{U^w}$ is weakly compact can be removed in Theorem 3.3 if we assume $F: \overline{U^w} \to CK(C)$ is weakly upper semicontinuous.

Proof. The proof is exactly the same as in [14]; for completeness we include it here. Suppose F has no fixed points in $\partial_C U$ (otherwise we are finished). Let $E = (E, w)$ (the space E endowed with the weak topology) and

$$H = \{x \in \overline{U^w} : x \in \lambda F x \text{ for some } \lambda \in [0,1]\}.$$

As in [14] we have $H \neq \emptyset$, with H weakly closed and weakly compact in $\overline{U^w}$. Also since $E = (E, w)$ is Tychonoff and since $H \cap \partial_C U = \emptyset$ there exists a weakly continuous function $\mu: \overline{U^w} \to [0,1]$ with $\mu(H) = 1$ and $\mu(\partial_C U) = 0$. Let

$$J(x) = \begin{cases} \mu(x)F(x) \equiv G(x), & x \in \overline{U^w} \\ \{0\}, & x \in C \backslash \overline{U^w}. \end{cases}$$

In [14] we showed that $F : \overline{U^w} \to CK(C)$ is weakly upper semicontinuous. We now show that $J : C \to CK(C)$ is weakly upper semicontinuous. To see this let V be a weakly closed subset of C. Then

$$\begin{aligned}&\{x \in C : J(x) \cap V \neq \emptyset\}\\ &\qquad = \{x \in \overline{U^w} : G(x) \cap V \neq \emptyset\} \cup \{x \in C\backslash\overline{U^w} : \{0\} \cap V \neq \emptyset\}.\end{aligned}$$

There are two cases to consider, namely $0 \in V$ and $0 \notin V$.

Case (i). $0 \notin V$.
Then

$$\{x \in C : J(x) \cap V \neq \emptyset\} = \{x \in \overline{U^w} : G(x) \cap V \neq \emptyset\} = G^{-1}(V),$$

which is weakly closed in $\overline{U}$ (so weakly closed in C) since $G: \overline{U^w} \to CK(C)$ is weakly upper semicontinuous.

Case (ii). $0 \in V$.
Then

$$\begin{aligned}\{x \in C : J(x) \cap V \neq \emptyset\} &= \{x \in \overline{U^w} : G(x) \cap V \neq \emptyset\} \cup (C\backslash\overline{U^w})\\ &= \{x \in \overline{U^w} : G(x) \cap V \neq \emptyset\} \cup (C\backslash(\operatorname{int}_C U));\end{aligned}$$

here $\operatorname{int}_C U$ denotes the weak interior of U in C. To see this we need only show that if $x \in C\backslash(\operatorname{int}_C U)$ then

$$x \in \{y \in \overline{U^w} : G(y) \cap V \neq \emptyset\} \cup (C\backslash\overline{U^w}).$$

This is immediate if $x \in C\backslash(\operatorname{int}_C U)$ and $x \in C\backslash\overline{U^w}$ also. So it remains to consider the case when $x \in C\backslash(\operatorname{int}_C U)$ and

$$x \notin C\backslash\overline{U^w} = C\backslash((\operatorname{int}_C U) \cup \partial_C U) = (C\backslash(\operatorname{int}_C U)) \cap (C\backslash\partial_C U),$$

i.e., when $x \in C\backslash(\operatorname{int}_C U)$ and $x \notin (C\backslash\partial_C U)$. In this case $G(x) = 0 \in V$, so $x \in \{y \in \overline{U^w} : G(y) \cap V \neq \emptyset\}$. Consequently

$$\{x \in C : J(x) \cap V \neq \emptyset\} = G^{-1}(V) \cup (C\backslash(\operatorname{int}_C U)),$$

which is weakly closed in C.

Thus $J: C \to CK(C)$ is weakly upper semicontinuous. In addition the argument in [14] guarantees that J is an α w–contractive map. Now [14, Theorem 2.2] guarantees that there exists $x \in C$ with $x \in J(x)$. If $x \in C\backslash\overline{U^w}$ then $x \in J(x) = \{0\}$, which is a contradiction since $0 \notin \partial_C U$ (see (3.2) with $\lambda = 0$). Thus $x \in U$ so $x \in \mu(x)F(x)$. As a result $x \in H$, so $\mu(x) = 1$ and $x \in F(x)$. ∎

Finally we present a Furi–Pera theorem for weakly sequentially upper semicontinuous maps.

THEOREM 3.4. *Let $E = (E, \|\cdot\|)$ be a separable and reflexive Banach space; C and Q are closed bounded convex subsets of E with $Q \subseteq C$ and $0 \in Q$. Assume $F: Q \to CK(C)$ is a weakly sequentially upper semicontinuous (and weakly compact) map and suppose there exists $\delta > 0$ with*

$$\Omega_\delta = \left\{x \in E : d(x, Q) \leq \delta\right\} \subseteq C;$$

here $d(x, y) = \|x - y\|$. In addition assume the following condition is satisfied:

$$\begin{array}{l} \textit{for any } \epsilon \leq \delta, \textit{ if } \{(x_j, \lambda_j)\}_1^\infty \textit{ is a sequence in } Q \times [0,1] \\ \textit{with } x_j \rightharpoonup x \in \partial_{\Omega_\epsilon} Q \textit{ and } \lambda_j \to \lambda \textit{ and if } x \in \lambda F(x) \\ \textit{for } 0 \leq \lambda < 1, \textit{ then } \{\lambda_j F(x_j)\} \subseteq Q \textit{ for } j \textit{ sufficiently} \\ \textit{large; here } \partial_{\Omega_\epsilon} Q \textit{ denotes the weak boundary of } Q \\ \textit{relative to } \Omega_\epsilon \textit{ and } \rightharpoonup \textit{ denotes weak convergence.} \end{array} \tag{3.3}$$

Then F has a fixed point in Q.

Remark 3.7. A special case of (3.3) is the following condition:

$$\begin{array}{l} \text{if } \{(x_j, \lambda_j)\}_1^\infty \text{ is a sequence in } Q \times [0,1] \text{ with } x_j \rightharpoonup x \text{ and} \\ \lambda_j \to \lambda \text{ and if } x \in \lambda F(x) \text{ for } 0 \leq \lambda < 1, \text{ then } \{\lambda_j F(x_j)\} \subseteq Q \\ \text{for } j \text{ sufficiently large.} \end{array} \tag{3.4}$$

Proof Let $r: E \to Q$ be a weakly continuous retraction (see [12] for existence) and let

$$B = \{x \in E : x \in Fr(x)\}.$$

The argument in [12] guarantees that $B \neq \emptyset$ is weakly closed and weakly compact. It remains to show $B \cap Q \neq \emptyset$. To do so we argue by contradiction. Suppose $B \cap Q = \emptyset$. Then since Q is weakly compact and B is weakly closed we have from [6, p. 65] that

$$d(B, Q) = \inf\{\|x - y\| : x \in B, y \in Q\} > 0.$$

Thus there exists $\epsilon \leq \delta$ with $B \cap \Omega_\epsilon = \emptyset$; here Ω_ϵ is as described above. Notice Ω_ϵ is weakly compact since Ω_ϵ is closed and convex (so weakly closed) and bounded (in the norm topology). Also since E is separable we know from [3] that the weak topology on Ω_ϵ is metrizable; let $d^\star$ denote the metric. For $i \in \{1, 2, \ldots\}$ let

$$U_i = \left\{x \in \Omega_\epsilon : d^\star(x, Q) < \frac{\epsilon}{i}\right\}.$$

Fix $i \in \{1, 2, \ldots\}$. Now U_i is $d^\star$-open in Ω_ϵ, so U_i is weakly open in Ω_ϵ. Also

$$\overline{U_i^w} = \overline{U_i^{d^\star}} = \left\{x \in \Omega_\epsilon : d^\star(x, Q) \leq \frac{\epsilon}{i}\right\}$$

and

$$\partial_{\Omega_\epsilon} U_i = \left\{ x \in \Omega_\epsilon : d^\star(x, Q) = \frac{\epsilon}{i} \right\}.$$

Now $B \cap \Omega_\epsilon = \emptyset$ and Theorem 3.3 (applied with $U = U_i$, $Q = \overline{U_i^w}$, $C = \Omega_\epsilon$, and $F = Fr$; note (3.2) cannot hold) guarantees that there exists $\lambda_i \in [0, 1)$ and $y_i \in \partial_{\Omega_\epsilon} U_i$ with $y_i \in \lambda_i Fr(y_i)$. We can do this argument for each $i \in \{1, 2, \ldots\}$. Notice in particular since $y_i \in \partial_{\Omega_\epsilon} U_i$ that

$$\left\{ \lambda_i Fr(y_i) \right\} \not\subseteq Q \qquad \text{for each } i \in \{1, 2, \ldots\}. \tag{3.5}$$

Now look at

$$D = \{x \in E : x \in \lambda Fr(x) \text{ for some } \lambda \in [0, 1]\}.$$

The argument in [14] guarantees that D is weakly compact (so weakly sequentially compact by the Eberlein–Šmulian theorem). This together with

$$d^\star(y_j, Q) = \frac{\epsilon}{j}, \quad |\lambda_j| \le 1 \qquad \text{for } j \in \{1, 2, \ldots\}$$

implies that we may assume without loss of generality that

$$\lambda_j \to \lambda^\star \qquad \text{and} \qquad y_j \rightharpoonup y^\star \in \overline{Q^w} \cap \overline{(\Omega_\epsilon \backslash Q)^w} = \partial_{\Omega_\epsilon} Q.$$

Also since $y_j \in \lambda_j Fr(y_j)$ we have that $y^\star \in \lambda^\star Fr(y^\star)$ (recall [14] that $Fr: C \to CK(C)$ is weakly upper semicontinuous). If $\lambda^\star = 1$ then $y^\star \in Fr(y^\star)$ which contradicts $B \cap Q = \emptyset$. Thus $0 \le \lambda^\star < 1$. But in this case (3.3), with

$$x_j = r(y_j) \qquad \text{and} \qquad x = y^\star = r(y^\star),$$

implies $\{\lambda_j Fr(y_j)\} \subseteq Q$ for j sufficiently large. This contradicts (3.5). Thus $B \cap Q \neq \emptyset$; i.e., there exists $x \in Q$ with $x \in Fr(x) = F(x)$. ∎

Remark 3.8. Usually in applications (see [12]) we take

$$C = \overline{\text{co}}(\overline{F(Q)^w} \cup \Omega_1).$$

Recall $\overline{F(Q)^w}$ and Ω_1 are weakly compact and as a result C is weakly compact by the Krein–Šmulian theorem [3, p. 434].

REFERENCES

1. N. Aronszajn and P. Panitchpakdi, Extensions of uniformly continuous transformations and hyperconvex metric spaces, *Pacific J. Math.* **6** (1956), 405–439.
2. J. B. Baillon, Nonexpansive mappings and hyperconvex spaces, in “Fixed Point Theory and its Applications” (R. F. Brown, Ed.), Contempory Math., Vol. 72, pp. 11–19, Am. Math. Soc., providence, 1998.
3. N. Dunford and J. T. Schwartz, “Linear Operators: Part I, General Theory,” Interscience, New York, 1985.
4. R. Espinola–Garcia, Darbo–Sadovski’s theorem in hyperconvex spaces, *Rend. Circ. Mat. Palermo, Ser. II Supp.* **40** (1996), 129–137.
5. P. M. Fitzpatrick and W.V. Petryshyn, Fixed point theorems for multivalued noncompact acyclic mappings, *Pacific J. Math.* **54** (1974), 17–23.
6. K. Floret, Weakly compact sets, *Lecture Notes Math.* **801** (1980), 1–123.
7. M. Furi and P. Pera, A continuation method on locally convex spaces and applications to ordinary differential equations on noncompact intervals, *Ann. Polon. Math.* **47** (1987), 331–346.
8. K. Goebel and W. A. Kirk, “Topics in Metric Fixed Point Theory,” Cambridge Univ. Press, Cambridge, UK, 1990.
9. W. A. Kirk and S. S. Shin, Fixed point theorems in hyperconvex spaces, *Houston J. Math.* **28** (1997), 175–188.
10. D. O’Regan, Some fixed point theorems for concentrative mappings between locally convex linear topological spaces, *Nonlinear Anal.* **27** (1996), 1437–1446.
11. D. O’Regan, A fixed point theorem for weakly condensing operators, *Proc. Roy. Soc. Edinburgh Sect. A* **126** (1996), 391–398.
12. D. O’Regan, A continuation method for weakly condensing operators, *Z. Anal. Anwendungen* **15** (1996),565–578.
13. D. O’Regan, A note on fixed point theorems in shells of Banach spaces for weakly condensing operators, *Z. Anal. Anwendungen* **16** (1997), 851–856.
14. D. O’Regan, Fixed point theory for weakly contractive maps with applications to operator inclusions in Banach spaces relative to the weak topology, *Z. Anal. Anwendungen* **17** (1998), 281–296.
15. D. O’Regan, Fixed point theory for weakly sequentially continuous mappings, *Math. Comput. Modelling* **27**, No. 5 (1998), 1–14.
16. D. O’Regan, Fixed point theory for closed multifunctions, *Arch. Math.* **34** (1998), 191–197.
17. D. O’Regan, Weakly contractive zero epi maps, *Math. Comput. Modelling* **30**, No. 7/8 (1999), 21–25.